Stationary ARMA Model

Class 9

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* This lecture note is written based on Professor Chang-Jin Kim's lecture note.

Introduction: ARMA Models

Approximation of Wold Representation

$$Y_t = \mu + e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

- There are infinite number of parameters to estimate to analyze impulse-response. $(\psi_1, \psi_2, \cdots)$
- BTW, if $\psi_1 = \phi$, $\psi_2 = \phi^2$, $\psi_3 = \phi^3$, $\cdots$, $\psi_j = \phi^j$, $\cdots$, then what we should estimate is just one, ϕ.
- Actually, the above restrictions imply the following simple model:

$$Y_t = \delta + \phi Y_{t-1} + e_t$$

  - Iterative substitution: $Y_t = \delta \left(1 + \phi + \phi^2 + \cdots + \phi^{j-1}\right) + \phi^j Y_{t-j} + e_t + \phi e_{t-1} + \cdots + \phi^j e_{t-j}$
  - $|\phi| < 1$: ϕ^j converges to zero when $j \to \infty$.
  - $j \to \infty$: $Y_t = \frac{\delta}{1-\phi} + e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^j e_{t-j} + \cdots$

1. AR(p) process (Auto-Regressive process of order p)

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-1} + \cdots + \phi_p Y_{t-p} + e_t$$

- We can transform the above process to Wold representation using iterative substitution.
- While Wold representation has infinite components, AR(p) process has finite components.
- The specific model of AR process depends on p $\rightarrow$ In the AR(p) process, Y_t is explained by its previous p terms.

2. MA(q) process (Moving-Average process of order q)

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$$

- Note that the shocks before time q do not affect Y_t on the above process.
- That is, in the MA(q) process, Y_t is explained by the current shock and previous q shocks.
- MA(q) process is a sort of approximation of Wold representation because it ignores very small θ_{q+j} for $j \geq 1$.

3. ARMA(p, q) process

$$Y_t = \underbrace{\delta + \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p}}_{AR(p)} + \underbrace{e_t + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q}}_{MA(q)}$$

- Mixed process: ARMA(p, q) = AR(p) + MA(q)
- In that sense, AR(p) = ARMA (p, 0) and MA(q) = ARMA(0, q)
- Also, we can transform the above process into Wold representation using iterative substitution.

Stationary AR(1) Process

Stationary AR(1) Process

$$Y_t = \delta + \phi Y_{t-1} + e_t, \quad e_t \sim iidN\left(0, \sigma^2\right)$$

1. Expectation

$$E\left(Y_t\right) = \delta + \phi E\left(Y_{t-1}\right) + E\left(e_t\right)$$

- Stationarity: Time-invariant expectation $i.e.$ $E\left(Y_t\right) = E\left(Y_{t-1}\right)$
- White noise: $E\left(e_t\right) = 0$

$$\begin{aligned} E\left(Y_t\right) &= \delta + \phi E\left(Y_t\right) \\ \rightarrow \quad (1-\phi) E\left(Y_t\right) &= \delta \\ \rightarrow \quad E\left(Y_t\right) &= \frac{\delta}{1-\phi} \end{aligned}$$

Stationary AR(1) Process [cont'd]

2. Variance

$$Var\left(Y_t\right) = Var\left(\delta\right) + \phi^2 Var\left(Y_{t-1}\right) + Var\left(e_t\right) + 2Cov\left(Y_{t-1}, e_t\right)$$

- Stationarity: Time-invariant variance *i.e.* $Var\left(Y_t\right) = Var\left(Y_{t-1}\right)$
- White noise: $Var\left(e_t\right) = \sigma^2$

$$\begin{aligned} Var\left(Y_t\right) &= \phi^2 Var\left(Y_t\right) + \sigma^2 \\ \rightarrow \quad \left(1-\phi^2\right) Var\left(Y_t\right) &= \sigma^2 \\ \rightarrow \quad Var\left(Y_t\right) &= \frac{\sigma^2}{1-\phi^2} \end{aligned}$$

Stationary AR(1) Process [cont'd]

3. Auto-Covariance (Auto-Correlation)

- Additional assumption: $\delta = 0$
  - Since we want to see a dynamics of Y_t, the constant δ does not harm a result.
  - Note that the constant δ matters for the first moment, not the second moment.
- Auto-Covariance:

$$\gamma_j = Cov\left(Y_t, Y_{t-j}\right) = E\left(Y_t Y_{t-j}\right)$$

Stationary AR(1) Process [cont'd]

- Here, $E\left(Y_t Y_{t-j}\right) = \gamma_j$, $E\left(Y_{t-1} Y_{t-j}\right) = \gamma_{j-1}$ and $E\left(e_t Y_{t-j}\right) = 0$:

$$\gamma_j = \phi \gamma_{j-1}$$

- Since auto-correlation ρ_j is defined by γ_j / γ_0:

$$\rho_j = \phi \rho_{j-1}$$

Stationarity Condition of AR(1) Model

$$\rho_0 = 1$$
$$\rho_1 = \phi$$
$$\rho_2 = \phi^2$$
$$\rho_3 = \phi^3$$
$$\vdots$$
$$\rho_\infty = \phi^\infty$$

- Recall, stationarity requires $\rho_\infty = \phi^\infty \to 0$!!

$$\therefore \text{Stationary condition is } |\phi| < 1$$

Impulse-Response Analysis

- Recall that we have the following under the stationary condition:

$$\frac{\partial Y_{t+j}}{\partial e_t} = \frac{\partial Y_t}{\partial e_{t-j}}$$

- In order to analyze impulse-response, we need the Wold representation.
- We can easily transform AR(1) model into the Wold representation:

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ &= \phi \left(\phi Y_{t-2} + e_{t-1} \right) + e_t = \phi^2 Y_{t-2} + e_t + \phi e_{t-1} \\ &\vdots \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^j e_{t-j} + \cdots \end{aligned}$$

- Therefore, we know that:

$$\frac{\partial Y_{t+j}}{\partial e_t} = \frac{\partial Y_t}{\partial e_{t-j}} = \phi^j$$

Stationary AR(2) Process

Stationary AR(2) Process

$$Y_t = \delta + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t, \qquad e_t \sim iidN\left(0, \sigma^2\right)$$

1. Expectation

$$\begin{aligned}
E\left(Y_t\right) &= \delta + \phi_1 E\left(Y_{t-1}\right) + \phi_2 E\left(Y_{t-2}\right) + E\left(e_t\right) \\
\rightarrow \quad E\left(Y_t\right) &= \delta + \phi_1 E\left(Y_t\right) + \phi_2 E\left(Y_t\right) \\
\rightarrow \quad \left(1 - \phi_1 - \phi_2\right) E\left(Y_t\right) &= \delta \\
\rightarrow \quad E\left(Y_t\right) &= \frac{\delta}{1 - \phi_1 - \phi_2}
\end{aligned}$$

Stationary AR(2) Process [cont'd]

2. Variance

- To earn the variance of Y_t we should compute a covariance between Y_{t-1} and Y_{t-2}, which requires complicated algebra.

- Just let the variance of Y_t be γ_0 (without computation).

$$Var\left(Y_t\right) = \gamma_0$$

Stationary AR(2) Process [cont'd]

3. Auto-Covariance (Auto-Correlation)

- Again, additional assumption: $\delta = 0$

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \\
\rightarrow \quad Y_t Y_{t-j} &= \left(\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t\right) Y_{t-j} \\
&= \phi_1 Y_{t-1} Y_{t-j} + \phi_2 Y_{t-2} Y_{t-j} + e_t Y_{t-j}
\end{aligned}
$$

- Taking expectation operator:

$$
\begin{aligned}
E\left(Y_t Y_{t-j}\right) &= \phi_1 E\left(Y_{t-1} Y_{t-j}\right) + \phi_2 E\left(Y_{t-2} Y_{t-j}\right) + E\left(e_t Y_{t-j}\right) \\
\rightarrow \quad \gamma_j &= \phi_1 \gamma_{j-1} + \phi_2 \gamma_{j-2} \\
(\rightarrow \quad \rho_j &= \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2})
\end{aligned}
$$

- Stationarity when $\gamma_j \rightarrow 0$ (or $\rho_j \rightarrow 0$) as $j \rightarrow \infty$.

Impulse-Response Analysis: State-Space Form

- Note that the impulse-response analysis in AR(1) was simple: $\frac{\partial Y_{t+j}}{\partial e_t} = \phi^j$
- **Idea**: If we transform AR(2) to a form of AR(1), the impulse-response analysis would be simple!
  - Define $\tilde{Y}_t = \begin{pmatrix} Y_t \\ Y_{t-1} \end{pmatrix}$, then $\tilde{Y}_{t-1} = \begin{pmatrix} Y_{t-1} \\ Y_{t-2} \end{pmatrix}$

Impulse-Response Analysis: State-Space Form [cont'd]

$$\begin{cases} Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t & : \text{AR}(2) \\ Y_{t-1} = Y_{t-1} \end{cases}$$

$$\rightarrow \begin{pmatrix} Y_t \\ Y_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ Y_{t-2} \end{pmatrix} + \begin{pmatrix} e_t \\ 0 \end{pmatrix}$$

$$\rightarrow \tilde{Y}_t = F \cdot \tilde{Y}_{t-1} + \tilde{e}_t \quad : \text{AR}(1) \text{ form}$$

Impulse-Response Analysis: State-Space Form [cont'd]

- Likewise, we can transform not only AR(2) but also AR(3), AR(4), $\cdots$, AR(p) into AR(1) form.
- We call the AR(1) form of AR(p) **State-Space form**.
  - *(optional)* A state-space model consists of a transition equation and a measurement equation.
    - Transition equation describes the evolution of the state vector over time.
    - Measurement equation relates the observed data to the state vector.
    - So, rigorously speaking, the AR(1) representation of AR(p) is the transition equation of the whole state-space representation.
- Now we can transform the state-space form of AR(2) to Wold representation.

$$\tilde{Y}_t = \tilde{e}_t + F\tilde{e}_{t-1} + F^2\tilde{e}_{t-2} + \cdots + F^j\tilde{e}_{t-j} + \cdots$$

Impulse-Response Analysis: State-Space Form [cont'd]

- What we want to know is the impact of e_{t-j} on Y_t.

$$\begin{pmatrix} Y_t \\ Y_{t-1} \end{pmatrix} = \begin{pmatrix} e_t \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e_{t-1} \\ 0 \end{pmatrix} + \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} e_{t-2} \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix}^j \begin{pmatrix} e_{t-j} \\ 0 \end{pmatrix} + \cdots$$

- Therefore,

$$\frac{\partial Y_{t+j}}{\partial e_t} = \frac{\partial Y_t}{\partial e_{t-j}} = F_{11}^j \quad : (1,1) \text{ element of } F^j$$

Stationarity Condition of AR(2): Eigenvalue

- Note that ϕ in AR(1) $Y_t = \phi Y_{t-1} + e_t$ determines the persistence of shock.
- Likewise, there exists "something" that determines persistence in AR(2), which is related to ***eigenvalues*** of the matrix F.
- Please refer to your materials (or textbooks or Google materials) for ***Mathematics for Economics*** or ***Linear Algebra*** to understand *eigenvalues*, *eigenvectors*, *characteristic equation*, and *diagonalization*.
- We will skip the detailed mathematics for this course, and instead will take the results and will learn the shortcut obtaining eigenvalues of F.

Stationarity Condition of AR(2): Eigenvalue [cont'd]

- We can diagonalize the matrix F by:

$$F = C \cdot \Lambda \cdot C^{-1}$$

  - where C is the matrix consisting of eigenvectors and Λ is a diagonal matrix that has distinct eigenvalues.
  - According to the properties of diagonalization, $F^j = C \cdot \Lambda^j \cdot C^{-1}$.

- Therefore,

$$F^j = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{pmatrix} \begin{pmatrix} c_{11}^* & c_{12}^* \\ c_{21}^* & c_{22}^* \end{pmatrix}$$

$$\Longrightarrow \quad \frac{\partial Y_{t+j}}{\partial e_t} = F_{11}^j = c_{11} c_{11}^* \lambda_1^j + c_{12} c_{21}^* \lambda_2^j$$

Stationarity Condition of AR(2): Eigenvalue [cont'd]

- Stationarity condition requires F_{11}^{j} converges zero as j goes to infinity.
  - Note that $c_{11}c_{11}^{*} + c_{12}c_{21}^{*} = 1$ ($\because$ properties of inverse-matrix)
  - Thus, F_{11}^{j} is the *weighted average* of λ_1^j and λ_2^j.
- **Stationary condition**

$$|\lambda_1| < 1, \quad |\lambda_2| < 1$$

  - That is, ϕ in AR(1) is corresponding to λ_1, λ_2 in AR(2).
  - The eigenvalues determine the persistent of AR(2).
  - We can apply the above condition to AR(p) *i.e.* the stationary condition of AR(p) is $|\lambda_1| < 1$, $|\lambda_2| < 1$, $|\lambda_3| < 1$, $\cdots$, $|\lambda_p| < 1$.

Stationarity Condition of AR(2): Eigenvalue [cont'd]

- Then, how to get eigenvalue of the matrix F?
  - We should derive the ***characteristic equation*** to obtain eigenvalues.
  - And we can derive the characteristic equation easily from the autocorrelation function (without any complicated procedure).
- Autocorrelation function of AR(2)

$$\begin{aligned} \rho_j &= \phi_1\rho_{j-1} + \phi_2\rho_{j-2} \\ \rightarrow \quad \rho_j &- \phi_1\rho_{j-1} - \phi_2\rho_{j-2} = 0 \end{aligned}$$

- Characteristic equation of AR(2)

$$\lambda^2 - \phi_1\lambda - \phi_2 = 0$$

MA(q) Process

MA(1) Process

$$Y_t = \mu + e_t + \theta e_{t-1}, \quad e_t \sim iidN\left(0, \sigma^2\right)$$

1. **Expectation**

$$E\left(Y_t\right) = E\left(\mu\right) + E\left(e_t\right) + \theta E\left(e_{t-1}\right) = \mu$$

- Note that we do not need any assumption of stationarity.

2. **Variance**

$$Var\left(Y_t\right) = Var\left(e_t\right) + \theta^2 Var\left(e_{t-1}\right) = \left(1 + \theta^2\right)\sigma^2$$

- Again, we do not need any assumption of stationarity.

3. Auto-Covariance (Auto-Correlation)

- Additional assumption: $\mu = 0$
- Auto-Covariance with time difference 1:

$$\begin{aligned}
\gamma_1 &= Cov\left(Y_t, Y_{t-1}\right) = E\left(Y_t Y_{t-1}\right) \\
&= E\left[\left(e_t + \theta e_{t-1}\right)\left(e_{t-1} + \theta e_{t-2}\right)\right] \\
&= E\left(e_t e_{t-1} + \theta e_t e_{t-2} + \theta e_{t-1}^2 + \theta^2 e_{t-1} e_{t-2}\right) \\
&= \theta E\left(e_{t-1}^2\right) \\
&= \theta \sigma^2
\end{aligned}$$

MA(1) Process [cont'd]

- How about auto-covariance with time difference 2:

$$\begin{aligned}\gamma_2 &= Cov\left(Y_t, Y_{t-2}\right) = E\left(Y_t Y_{t-2}\right) \\ &= E\left[\left(e_t + \theta e_{t-1}\right)\left(e_{t-2} + \theta e_{t-3}\right)\right] \\ &= E\left(e_t e_{t-2} + \theta e_t e_{t-3} + \theta e_{t-1} e_{t-2} + \theta^2 e_{t-1} e_{t-3}\right) \\ &= 0\end{aligned}$$

- Likewise, the auto-covariances with time difference greater than 2 are zero!
- Therefore, we know that

$$\begin{cases}\rho_1 \neq 0 \\ \rho_2 = \rho_3 = \rho_4 = \cdots = 0\end{cases}$$

4. Impulse-response analysis

- MA(1) process is the Wold representation in itself.

$$\frac{\partial Y_{t+1}}{\partial e_t} = \frac{\partial Y_t}{\partial e_{t-1}} = \theta$$

$$\frac{\partial Y_{t+j}}{\partial e_t} = \frac{\partial Y_t}{\partial e_{t-j}} = 0 \quad \text{for } j \geq 2$$

4. Stationary condition

- MA(1) process is ALWAYS stationary without any condition!

MA(2) Process

$$Y_t = \mu + e_t + \theta_1 e_{t-1} + \theta e_{t-2}, \qquad e_t \sim iidN\left(0, \sigma^2\right)$$

1. **Expectation**

$$E\left(Y_t\right) = E\left(\mu\right) + E\left(e_t\right) + \theta_1 E\left(e_{t-1}\right) + \theta_2 E\left(e_{t-2}\right) = \mu$$

2. **Variance**

$$Var\left(Y_t\right) = Var\left(e_t\right) + \theta_1^2 Var\left(e_{t-1}\right) + \theta_2^2 Var\left(e_{t-2}\right) = \left(1 + \theta_1^2 + \theta_2^2\right)\sigma^2$$

MA(2) Process [cont'd]

3. Auto-Covariance (Auto-Correlation)

$$\begin{cases} \gamma_1 \neq 0, \ \gamma_2 \neq 0 \\ \gamma_3 = \gamma_4 = \gamma_5 = \cdots = 0 \end{cases}$$

- We can extend our result to MA(q) model.
- Then, MA(q) process is always stationary?
- Yes! MA(q) process is stationary only if q is **FINITE**!

Stationary ARMA(p, q) Model

Stationary ARMA Model

- So far, we have learned AR model and MA model.
  - We need the stationary condition for AR model, whereas MA model does not require any condition other than finite order.
- **ARMA model** is a mixed form of AR model and MA model $\rightarrow$ It is intuitive that AR part determines the stationarity of ARMA model.
  - For example, the stationary condition of ARMA(2,1) is equal to the stationary condition of AR(2) (or ARMA (2,0)).

Lag Operator

- **Lag Operator (L)**
  - In time series analysis, the lag operator L operates on an element of a time series to produce the previous element.
  - For example,

$$LY_t = Y_{t-1}$$

  - The lag operator can be raised to arbitrary integer powers so that

$$L^k Y_t = Y_{t-k}$$

  - For example,

$$L^2 Y_t = LLY_t = L(LY_t) = LY_{t-1} = Y_{t-2}$$
$$L^3 Y_t = LLLY_t = LL(LY_t) = L(LY_{t-1}) = LY_{t-2} = Y_{t-3}$$

AR Model in Lag Operator

- AR(1) model:

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ \rightarrow \quad Y_t - \phi L Y_t &= e_t \\ \rightarrow \quad (1 - \phi L)\, Y_t &= e_t \end{aligned}$$

  - Define $\phi(L) = 1 - \phi L$ which is called ***polynomial equation in lag operator***.

- AR(2) model:

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t \\ \rightarrow \quad Y_t - \phi_1 L Y_t - \phi_2 L^2 Y_t &= e_t \\ \rightarrow \quad \left(1 - \phi_1 L - \phi_2 L^2\right) Y_t &= e_t \\ \rightarrow \phi(L)\, Y_t &= e_t \quad \text{where}\, \phi(L) = 1 - \phi_1 L - \phi_2 L^2 \end{aligned}$$

Converting into Wold Representation

- Why is the lag operator useful?
- Note that our goal is converting a stationary process into a Wold representation.
- **Method 1**: Iterative substitution

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ &= \phi\left(\phi Y_{t-2} + e_{t-2}\right) + e_t \\ &= \phi^2 Y_{t-2} + e_t + \phi e_{t-2} \\ &\vdots \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots \end{aligned}$$

  - This method is not that difficult in the case of AR(1).
  - But we already know that AR(2) or higher order AR(p) are complicated in using iteration.

Converting into Wold Representation [cont'd]

- **Method 2**: Lag operator

$$\begin{aligned} Y_t &= \phi Y_{t-1} + e_t \\ \rightarrow \quad (1 - \phi L)\, Y_t &= e_t \\ \rightarrow \quad Y_t &= \frac{1}{1 - \phi L} e_t \end{aligned}$$

  - The stationary condition of AR(1) is $|\phi| < 1 \rightarrow$ If we treat it as $|\phi L| < 1$:

$$\frac{1}{1 - \phi L} = 1 + \phi L + (\phi L)^2 + (\phi L)^3 + \cdots$$

  - Then, the last equation becomes:

$$\begin{aligned} Y_t &= \left(1 + \phi L + (\phi L)^2 + (\phi L)^3 + \cdots\right) e_t \\ &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots \end{aligned}$$

Converting into Wold Representation [cont'd]

- We can generalize the method 2 with respect to AR(p):

$$
\begin{aligned}
Y_t &= \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t \\
\rightarrow \quad \phi(L) Y_t &= e_t \\
\rightarrow \quad Y_t &= \phi^{-1}(L) e_t
\end{aligned}
$$

ARMA Model in Lag Operator

- MA(1) model:

$$
\begin{aligned}
& Y_t = \mu + e_t + \theta e_{t-1} \\
\rightarrow \quad & Y_t = \mu + e_t + \theta L e_t \\
\rightarrow \quad & Y_t = \mu + (1 + \theta L)\, e_t \\
\rightarrow \quad & Y_t = \mu + \theta (L)\, e_t \ \text{ where } \theta (L) = 1 + \theta L
\end{aligned}
$$

- In general MA(q) model is $Y_t = \mu + \theta (L)\, e_t$
- Then, how about ARMA(p, q)?

$$
\phi (L)\, Y_t = \mu + \theta (L)\, e_t
$$

- As we discussed, ARMA model can be always converted into the Wold representation.

Stationary Condition Revisited

- Another usefulness of lag operator is regarding stationary condition.
- Recall, in the case of AR(2) model, the autocorrelation function is:

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$$

  - We can derive the characteristic equation directly from the above:

$$\lambda^2 - \phi_1 \lambda - \phi_2 = 0$$

  - Then we can solve to λ and get two solutions λ_1 and λ_2.
- The stationary condition of AR(2) is:

$$|\lambda_1| < 1, \quad |\lambda_2| < 1$$

Stationary Condition Revisited [cont'd]

- Note that the polynomial equation in lag operator for AR(2) is:

$$\phi\left(L\right) = 1 - \phi_1 L - \phi_2 L^2$$

- Suppose that we solve $\phi\left(L\right) = 0$ to L. What are the solutions?
  - Substitute L with $1/\lambda$, then

$$\begin{aligned} 1 - \phi_1 \left(\frac{1}{\lambda}\right) - \phi_2 \left(\frac{1}{\lambda}\right)^2 &= 0 \\ \rightarrow \ \ \lambda^2 - \phi_1 \lambda - \phi_2 &= 0 \end{aligned}$$

  - That is, the solutions of $\phi\left(L\right) = 0$ is the reciprocal of the solutions of characteristic equations.

$$L_1 = \frac{1}{\lambda_1}, \ \ L_2 = \frac{1}{\lambda_2}$$

- Therefore, we can rewrite the stationary condition by:

$$|\lambda_1| < 1, \; |\lambda_2| < 1 \quad \Longleftrightarrow \quad |L_1| > 1, \; |L_2| > 1$$

- Also we can generalize the above result onto ARMA(p, q)!

Box-Jenkin's Approach

Partial Autocorrelation Function

- **Question**: How to identify the order p and q for ARMA model?
  - In case of MA model, we can easily identify q from autocorrelation function.
  - However, AR model as well as ARMA model is difficult to identify the order just from the autocorrelation function.
  - Therefore, *partial autocorrelation* is suggested as a supplementary method.
- For example, suppose that we have a time series data Y_t that AR(1) model explains best.
  - But we don't know AR(1) model is the best one for Y_t ex ante.
  - So we consider all AR(p) models as candidates, and then test the significance of p.

Partial Autocorrelation Function [cont'd]

$$
\begin{aligned}
\text{AR}(1):\ & Y_t = \phi_{11} Y_{t-1} + e_t \\
\text{AR}(2):\ & Y_t = \phi_{21} Y_{t-1} + \phi_{22} Y_{t-2} + e_t \\
\text{AR}(3):\ & Y_t = \phi_{31} Y_{t-1} + \phi_{32} Y_{t-2} + \phi_{32} Y_{t-3} + e_t \\
& \vdots \\
\text{AR(j)}:\ & Y_t = \phi_{j1} Y_{t-1} + \phi_{j2} Y_{t-2} + \cdots + \phi_{jj} Y_{t-j} + e_t \\
& \vdots
\end{aligned}
$$

- Estimate each model and get $\hat{\phi}_{11}, \hat{\phi}_{22}, \hat{\phi}_{33}, \cdots, \hat{\phi}_{jj}, \cdots$.
  - $\hat{\phi}_{11}, \hat{\phi}_{22}, \hat{\phi}_{33}, \cdots, \hat{\phi}_{jj}, \cdots$ are called partial correlations.
  - Since the data is explained by AR(1) best, $\hat{\phi}_{11}$ will be significant and the others will not be significant.
- In general, if we find significant $\hat{\phi}_{jj}$, the model that explains data best would be AR(j).

Box-Jenkin's Approach to ARMA

- **Question**: How to identify the order p and q for ARMA model?
- **[Step 1]** Given data, draw the autocorrelation function and the partial autocorrelation function.
- **[Step 2]** Based on the ACF and the PACF, select the candidates of combination of p and q for ARMA (p, q).
- **[Step 3]** Estimate the coefficients of each candidate.
- **[Step 4]** Diagnostic test: *White noise test*
  - If the residual of a model does not pass the white noise test, the model will not be a good one.
- **[Step 5]** Select the best model using **AIC** (Akaike Information Criterion) or **BSC** (Bayes-Schwartz Criterion)
  - We will choose the model whose score is lower.